PreCalc Review Packet
For Students Entering AP Calculus (AB)

Directions

Read and complete the problems as assigned, showing appropriate work on separate paper. **The problems are due on Tues, Sept 3 as your entry ticket into the class.** If you have questions, feel free to email me. I am very eager to help motivated students who ask questions, and no question is considered dumb or too small. The suggested pacing gives you enough time on either side in case you have vacation plans. *If the material seems challenging, email me!*

jcagle@chapelgateacademy.org

Suggested Pacing	**Section**
June 25:	**Read** Linear Functions. **Do** Linear Functions Questions (pp 2-4)
June 27:	**Read** Functions. **Do** Functions Questions (pp 5-8)
June 29:	**Read** Polynomial Functions. **Do** Polynomial Functions Questions (pp 9-10)
Jul 2:	**Read** Exponential Functions. **Do** Exponential and Logarithmic Function Questions #1 – 4 (pp 11 – 13)
Jul 4:	**Read** Logarithmic Functions. **Do** Exponential and Logarithmic Function Questions #5 – 7 (pp 11 – 13)
Jul 6:	**Read** Trig Functions. **Do** Trig Function Questions #1-2 (pp 14 – 17)
Jul 9:	**Read** Trig Functions. **Do** Trig Function Questions #3 (pp 14 – 17)
Jul 11:	**Read** Trig Functions. **Do** Trig Function Questions #4-7 (pp 14 – 17)

Helpful hints

- Make sure that you do the reading and the "before starting the problem" self-checks. If you try to skip those, you will have a hard time. If you faithfully do them, your work will go better and faster. Trust me on this.
- Some problems are marked *No calculator*. Others are marked *Calculator*. Follow the directions in both cases.
- There are answers on p 18. Use those wisely: After working each problem, check yourself. If you are stuck, work backwards.

- The expectation is that you will walk into class on the first day knowing how to do 90% or more of the problems. If you find yourself needing help, email me sooner than later!!

Linear Functions

If we take any two points (x_1, y_1) and (x_2, y_2), the *slope* between those two points is given by

$$(1) \quad slope = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

In practical applications, the slope is interpreted as the *average rate of change on the interval* $[x_1, x_2]$.

A *linear function* is a function of the form

$$(2) \quad f(x) = mx + b$$

The reason we call these functions *linear* is that they have a constant rate of change. This is formalized in a theorem:

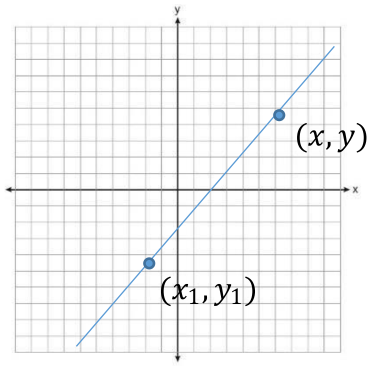
The Constant Rate Theorem ("Lines have a constant rate of change")

If $f(x) = mx + b$, the graph has a constant average rate of change of m between any two points.

Proof: Let $y = f(x) = mx + b$ be any linear function and pick any two x-values x_1 and x_2. Then the average rate of change of $f(x)$ will be

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{mx_2 - mx_1}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m$$

The linear function form $y = mx + b$ is often called the *slope-intercept form*. It is useful in graphing the linear function because the value b will be the y-intercept of the graph (see Ex. 1 in the problems).



Often, we have a situation where we know the slope *m* of a line and a point (x_1, y_1) on the line that is not the y-intercept.

If we pick any other point (x, y) on the line and compute slope, we get

$$m = \frac{y - y_1}{x - x_1}$$

The value *m* is guaranteed to be the same for all points (x,y) because of our Constant Rate Theorem. Rearranging, we get the *Point-Slope Form* of a linear function:

(3) $y - y_1 = m(x - x_1)$

Here are some important facts to know about linear functions.

- In geometry, you learned the important postulate "two points determine a line." This means that you can confirm the correctness of a linear equation by making sure that it correctly goes through two known points.
- Two lines with slopes m_1 and m_2 are *parallel* if $m_1 = m_2$. Two lines are *perpendicular* or *normal* if their slopes satisfy $m_1 m_2 = -1$.
- A *horizontal* line has a slope of 0 and an equation of the form $y=c$. *Proof*: The ycoordinates of all points on a horizontal line must be the same. Take two distinct points on a horizontal line $(a,c)$and (b,c). Plug into the slope equation to get $m = \frac{c - c}{b - a} = 0$.

  Now use the point-slope form: $y - c = 0(x-a) = 0$so that $y=c$. Done.
- A *vertical* line has no defined slope and an equation of the form $x=a$. Proof: a vertical line that goes through (a, b) will also go through (a, y) for any y, so that all points (a, y) will be on that vertical line. If we try to compute the slope between (a, b) and (a, y), we find that the denominator is 0, which is illegal.

Linear Function Questions

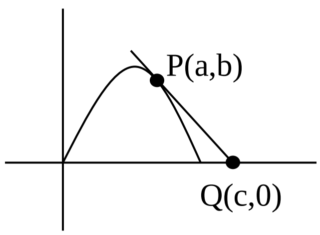
No Calculator

0. Self-check: Before getting started on problems, make sure that you can state (without looking) the following:

   The Slope-Intercept Form

The Point-Slope Form
The Constant Rate Theorem
The condition for two non-vertical lines to be parallel
The condition for two non-vertical lines to be perpendicular or normal.

1. Find the equation of the line that goes through the given points. Confirm your answers by checking that each point satisfies your function.
   a. Through (1,5) and (3,9)
   b. Through (0,0) and (2.8, 5.6)
   c. Through (6,1.5) and (6,8)
   d. The line shown in the graph below (your answer is in terms of a, b, and c)



2. Show that the y-intercept of a linear function $f(x) = mx + b$ is b. (Hint: what is the definition of "y-intercept"? Check the internet if needed).

3. What is the equation of the line with slope m through the point $(c, f(c))$?

(a) $y = m(x - f(c)) + c$
(b) $y = m(x - c) + f(c)$
(c) $y = m(x - c) - f(c)$
(d) $y = -m(x - f(c)) + c$
(e) $y = f(c)$

4. What is the equation of the *horizontal* line through the point (1,3)?

(a) $x=1$ (b) $y=1$ (c) $x=3$ (d) $y = 3$ (e) None of these

5. Which of the following is (are) true?

I. A horizontal line has no defined slope.
II. On a given line, any pair of points has the same slope as any other pair.
III. Given an point P(a,b) and a line l of slope m, the equation of the line normal (perpendicular) to l through (a,b) is given by $y - a = -\frac{1}{m}(x - b)$.

(a) I only (b) II only (c) I and III (d) II and III (e) I, II, and III

6. Show that

   a. The line $\frac{y-3}{x-4} = 5$ is parallel to the line through (1,6) and (3,16).

   b. The line through (a,0) and (0,b) can be written in the form $\frac{x}{a} + \frac{y}{b} = 1$.

7. A line has general form $4x + 2y = 1$. Write this in slope intercept form.

Calculator – Remember to use 4 places where needed!

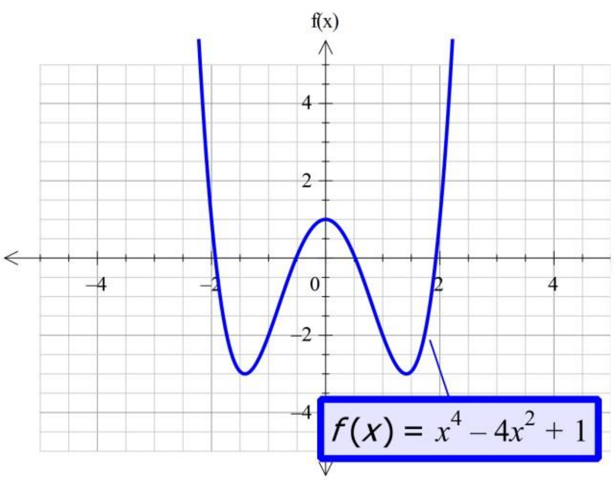
8. If a car is purchased for $15,000 at the beginning of 2012 and loses $1,700 of value per year, in what year will it become worthless? Create a linear function V(t) to represent the value of the car.

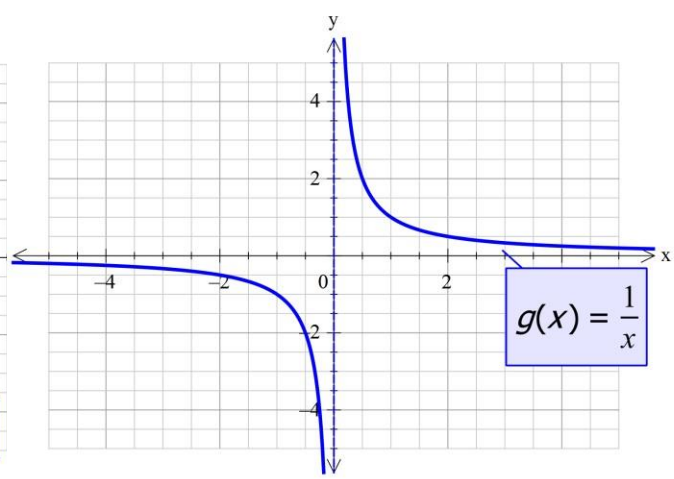
9. Water boils at 212°F (100°C) and freezes at 32°F (0°C). Using this information, create a linear equation C(F) that gives the temperature in Celsius for a given temperature in Fahrenheit.

Functions

A *function* is a relationship between a set called the *domain* and a set called the *range* such that every element in the domain corresponds to exactly one element in the range. Functions can be specified by an equation, a graph, a table of values, or a verbal description.

Definition: A function f(x) is *even* if $f(-x) = f(x)$. A function is *odd* if $f(-x) = -f(x)$





In the graphs above, $f(x)$ is even because $f(-x) = f(x)$. $g(x)$ is odd because $g(-x) = -g(x)$

A function f(x) is called *one-to-one* (usually written 1-1) if every element in the *range* corresponds to exactly one element in the domain. This is sometimes written as

Definition of 1-1: f(x) is 1-1 if $f(a) = f(b)$ implies that a = b.

The *inverse* of f(x) is denoted $f^{-1}(x)$ and has the property that it "cancels" f(x) like this:

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$. An inverse is found by swapping x's and y's. In class, we will prove the

Inverse Existence Theorem: $f(x)$ has an inverse iff. it is 1-1.

Here are some useful facts to know about functions.

- Standard Domain Assumption: If a function is given without a specified domain, you should assume that the domain is all reals except for illegal or physically unreasonable values. For example, the domain of $f(x) = \frac{1}{x}$ is $\{x: x < 0 \text{ or } x > 0\}$. The domain of the function " $L(w)$ =length of wire with weight w" is $\{w: w > 0\}$.
- A function is odd if and only if its graph is symmetric about the origin. A function is even if and only if its graph is symmetric about the y-axis.

Functions Problems

No Calculator

0. Before starting the problems, make sure that you can state without looking the following

The definition of function
The definition of an inverse function
The definitions of even and odd functions
The definition of 1-1
The Inverse Existence Theorem

1. Sketch the graphs of the following functions on a reasonable window size. Be sure to label axes and include window size.

$f(x) = 3x+5$

$g(x) = \sqrt{x+2}$

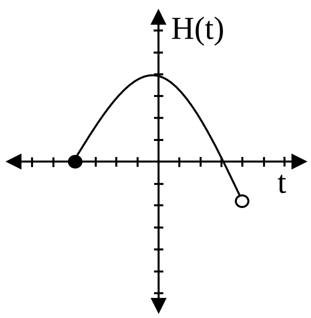
$A(r)$ is the area of the circle with radius r.

2. Find the domain and range of the following functions.

$p(x) = \dfrac{1}{x-2}$

$q(w) = \sqrt{1-w^2}$

$H(t)$ is defined by the graph



3. Even, odd, or neither? Justify your answer.

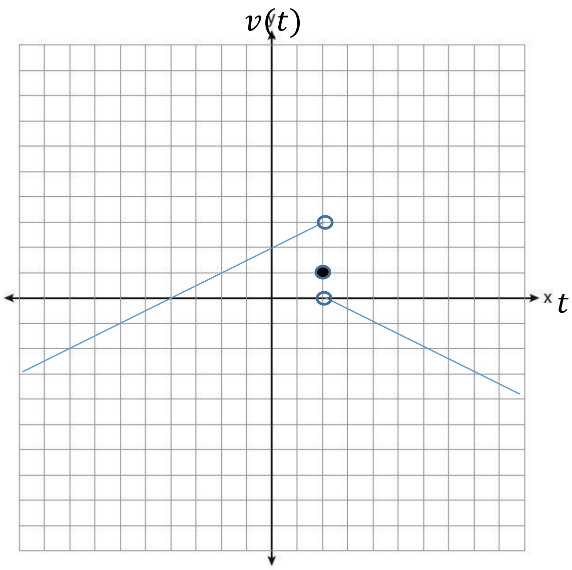
$g(x) = e^x - e^{-x}$

$h(x) = x^3 + 1$

$m(n) = \dfrac{1}{n^2 - 1}$

4. Sketch: $f(x) = \begin{cases} 3 + x & 0 \le x \le 3 \\ 2x & x > 3 \end{cases}$

5. The graph below consists of two straight lines and a single point. Write the piecewise formula for the function.



Calculator. Questions 6 and 9 are multiple-choice.

6. Which of the following is (are) an odd function?

   I. $f(x) = x^3 + 1$

   II. $g(x) = x^3 - x$

   III. $h(x) = \frac{x^3}{x^2 + 1}$

a. I only b. II only c. III only d. I and II e. II and III

7. Sketch in the window [-10,10]x[-3,3].

$$g(t) = \begin{cases} e^t & t \leq 0 \\ \ln(t+1) & t > 0 \end{cases}$$

8. Find the domain and range of $p(x) = \cos\left(\frac{1}{x}\right)$.

9. Which of the following have inverses? Circle all that apply and give the inverse function for each invertible function.

(a) $f(x) = 3x+5$

(b) $g(x) = 2x^2$

(c) $h(x) = \sin x$ (d) $p(□) = \cos□$ 0 □□□□

(e) $r(w) = e^w$

Polynomial Functions

A *polynomial function* is a function of the form $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ with $a_n \neq 0$. The "highest power" n is called the *degree* of the polynomial. Polynomials are useful as models for other, more complicated functions.

A polynomial is easiest to graph by factoring. The *factored form* of a polynomial is $P(x) = a_n(x - r_1)(x - r_2) \ldots .\, (x - r_n)$. The numbers r_i are called roots or zeros. The zeros are are not necessarily distinct. The number of times a zero is repeated is called the *multiplicity* of that zero. In the larger world of math, zeros can be complex numbers $a + ib$, but in this class, all of our zeros will be real numbers.

Ex.: $P(x) = 2\left(x - \frac{1}{2}\right)(x - 2)^2(x + 1)^3$ has three zeros $x = \frac{1}{2}$ with multiplicity 1, $x = 2$ with multiplicity 2, and $x = -1$ with multiplicity 3.

Ex.: $Q(x) = x^2 - 7$ can be factored as $(x - \sqrt{7})(x + \sqrt{7})$. What are its zeros and their multiplicities?

The virtue of writing a polynomial in factored form is that it allows us to sketch that polynomial using the "cut-bounce-hug" principle.

Cut-Bounce-Hug Principle

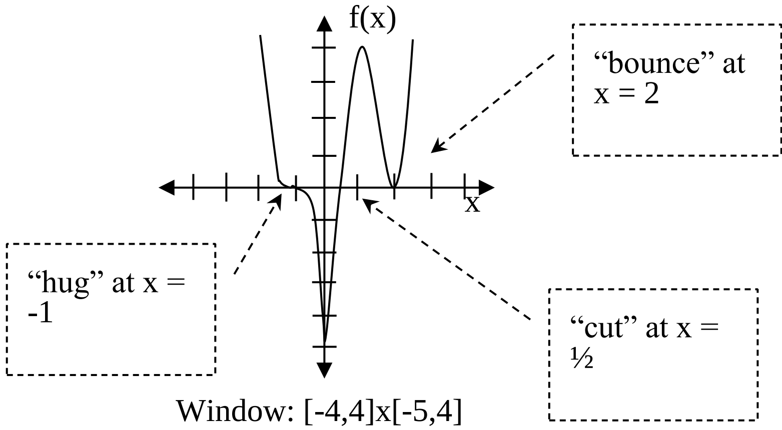
Suppose that a polynomial P(x) has a zero at x = r.

If the Multiplicity of r is	**Graph of P(x) will**
1	cut through the x-axis at x = r
even (2, 4, …)	bounce off the x-axis at x = r odd
but not 1 (3, 5, …)	"hug" the x-axis at x = r

Ex.: Sketch the graph of $f(x) = (2x-1)(x-2)^2(x+1)^3$.

Solution: We observe the zeros and their multiplicities: $x = \frac{1}{2}$ has multiplicity 1, $x = 2$ has multiplicity 2, and $x = -1$ has multiplicity 3. Furthermore, since $f(x) \approx 2x^6$ for very large positive and negative values of x, the graph will go up at both ends. In summary:

zero	**multiplicity**	**graph**
½	1	cuts
2	2	bounces
-1	3	hugs.



Window: [-4,4]x[-5,4]

Polynomial Function Problems

No Calculator

0. Before starting the problems, make sure that you can state without looking

   The definitions of *zero* and *multiplicity*
   The cut-bounce-hug principle

1. Is a linear function also a polynomial? Justify your answer.

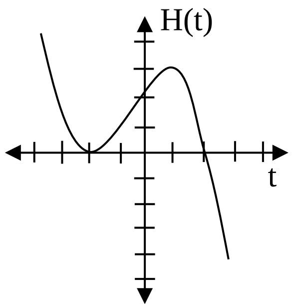
2. Sketch the graphs of the following polynomials.

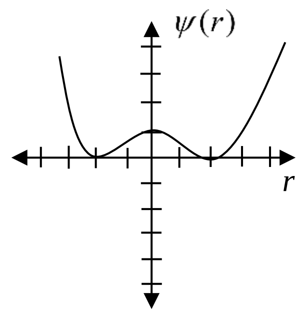
   $p(x) = (x-1)(x-2)(x-3)$

   $q(x) = -2(x-1)^2(x+2)$

   $r(x) = 3x^2 + 5x - 2$

3. The graphs of two polynomials are shown below. Give a possible equation for each of them.





4. Solve the following equations.

$$2x^2 - 11x + 12 = 0$$
$$x^3 - 2x^2 = 0$$

Calculator

5. Solve the following equations:

$$x^2 + 3.5x^4 = 2$$
$$x^2 - 1 = \ln x$$

Exponential Functions

An *exponential function* is one of the form $f(x) = Ab^x$, where A is any real number and b is any positive real. Here are some important facts about exponentials:

- Exponential Properties

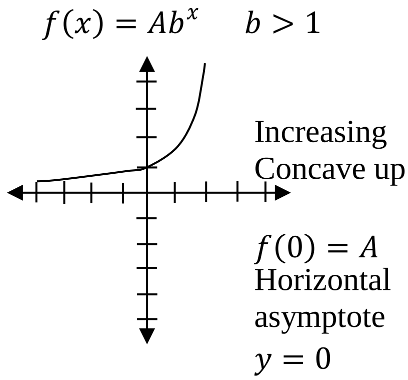
$$b^x b^y = b^{x+y}$$

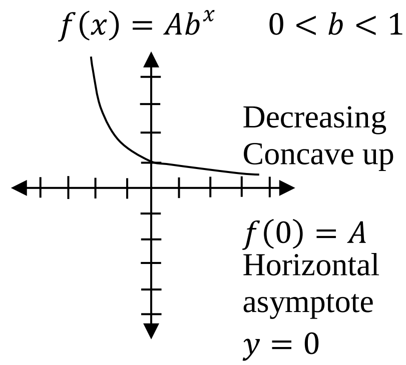
$$\frac{b^x}{b^y} = b^{x-y} \qquad (4)$$

$$(b^x)^y = b^{xy}$$

$$(ab)^x = a^x b^x$$

- The standard graph of $y = Ab^x$ has one of these shapes.





- The domain of $f(x) = Ab^x$ is (-□,+□) and the range is (0,□). Multiplying by A will change the range according to the usual transformation rules.
- The number b is called the *base* of the exponential, and A is the initial value or yintercept of the exponential.

Logarithmic Functions

The *logarithmic function* $f(x) = \log_b x$ is the inverse of an exponential. Accordingly, logs are used to "cancel" exponentials. Here are some important facts to know.

- The "inverse" property, frequently used to solve equations. $\log_b b^x = x$ (6)

  $b^{\log_b x} = x$
- Log Properties:

  $\log_b xy = \log_b x + \log_b y$

  $\log_b \frac{x}{y} = \log_b x - \log_b y$

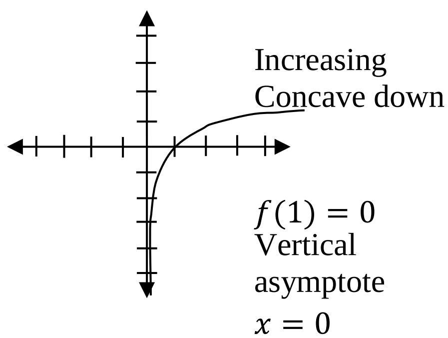
  (7) $\log_b x^y = y \log_b x$

  $\log_a x = \frac{\log_b x}{\log_b a}$

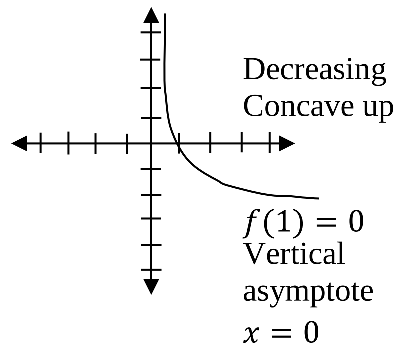
The two most common bases for logs are 10 and e. $\log_{10} x$ is written simply as $\log x$ on your calculator, and $\log_e x$ is universally written as $\ln x$, the "natural log" of x. We will learn this year why this log is "natural"

- The graph of a logarithmic function usually looks like this:

$f(x) = \log_b x \quad b > 1$



$f(x) = \log_b x \quad 0 < b < 1$



Exponential and Logarithmic Functions Problems

No Calculator. Question 15 is multiple-choice.

0. Before starting the problems, make sure you can state without looking

   The form of the exponential function
   The four exponential properties
   The four logarithmic properties
   Sketch the graphs of exponential and logarithmic functions, identifying intercept and asymptote.

1. Which of the following is (are) NOT an exponential function?

   I. $f(x) = e^{-x}$ II. $g(x) = □^x$ III. $h(x) = e^{x^2}$

a. I only   b. III only   c. I and III   d. I, II, and III   e. None of these.

2. Find the exponential equation that goes through the given points.

   (0, 4) and (2, 36)

(1, -1) and (3, -1/4)

3. Sketch graphs of the following:

   $f(x) = 2 \cdot 3^x$

   $g(x) = 4 - e^{-x}$

4. Solve the following equations.

   $4e^x = 3$

   $\frac{e^{3x}}{e^x} = 6$

   $e^{x^2} = 5$

5. Sketch $h(x) = \ln(x+3)$

6. Solve

   $\ln(x+1) = 3$

   $\log(x+2) - \log(x+1) = 1$ (Note: the base is 10)

Calculator

7. Solve the following:

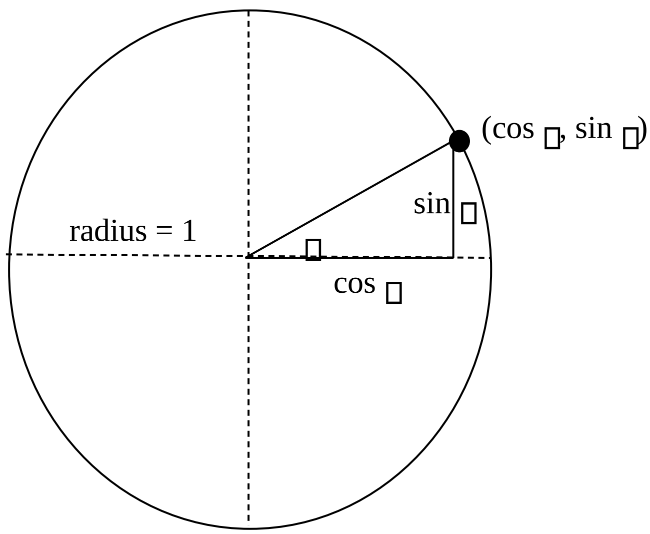
   $e^{-x^2} \square \frac{1}{2}$

   $4^x = 8x$

   $\ln(x) \square 0$

Trigonometric Functions

The trig functions all come from the unit circle. If you draw a central angle θ on the circle and form a right triangle with the radius as its hypoteneuse, then the hypoteneuse hits the circle at the point ($\cos\theta$, $\sin\theta$). That is: $\cos\theta$ is the x-coordinate and $\sin\theta$ is the y-coordinate on the unit circle.



From the definitions of sin θ and cos θ, we can define the following other trig functions:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

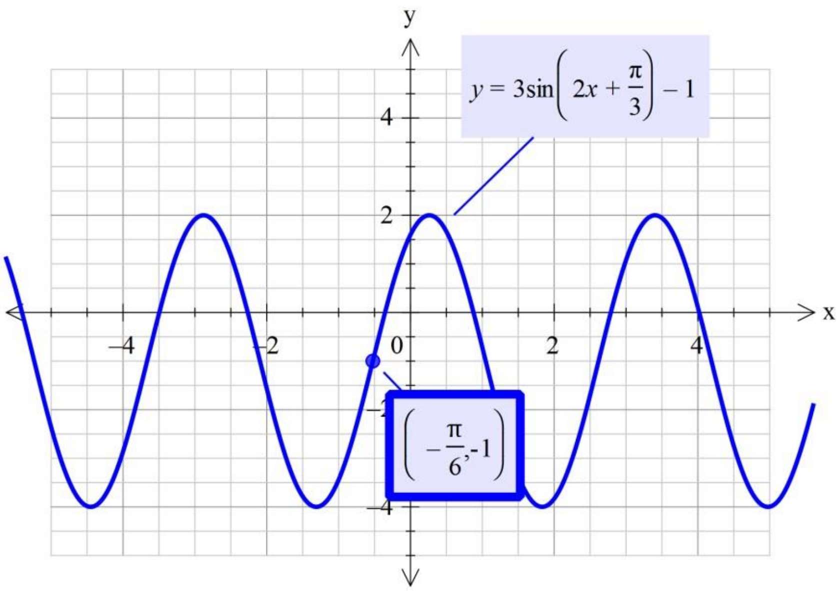
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

Graphing

It is important to be able to sketch graphs of trig functions. The graphs of sinusoids are defined by four parameters: amplitude A, period T, midline k, and horizontal shift h according to the equation $y = A\sin(\omega(x - h)) + k$

We can see all of these in this graph below.



$A = 3$
$\omega = 2$
$T = \frac{2\pi}{\omega} = \pi$
$k = -1$
$h = -\frac{\pi/3}{2} = -\frac{\pi}{6}$

The graphs of the other trig functions can be derived from the graphs of sine and cosine. For example, the graph of $y = \tan x$ can be derived from the fact that $\tan x = \frac{\sin x}{\cos x}$, so that the graph of tan x will have zeros at x = 0, π, 2π, 3π, … and poles (vertical asymptotes) at x = π/2, 3π/2, 5π/2, …

Identities Help Us Simplify and Solve

The other important skill is to be able to use trig identities to transform trig functions as needed. Here are the important trig IDs

Pythagorean IDs

Angle Addition IDs

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

^{2} $\cos^2$□$=1$ $\cos(x$□$y) = \cos x \cos y$□ $\sin x \sin y$ $\sin$ □$+$

$\sin(2x) = 2\sin x \cos x$

$\tan^2$□$+1= \sec^2$□ $\qquad$ $\cos(2x) = \cos^2 x - \sin^2 x$

$\cot^2$□$+1= \csc^2$□ $\qquad$ $= 2\cos^2 x - 1$

$=1 - 2\sin^2 x$

Ratio IDs

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Trig Function Problems

No Calculator

0. Before starting the problems, make sure that you can

   Draw the unit circle and mark the special angles on it.
   State the three Pythagorean identities
   State the five Ratio identities
   State the two Angle Addition identities for sine and cosine
   Sketch graphs of sine and cosine

1. Use the unit circle to find the values of the following. sin □ — cos — 5□

   4 3

   3□ □ □□ tan— sec□−

   —□

   4 □ 4 □

2. Use the unit circle to solve on the interval $[0,2\pi]$

$\sin x = \frac{1}{2}$ $\cos t = -\frac{1}{\sqrt{2}}$ $\cot x = -1$ $\csc x = 2$

$\sin^{-1}(x) = \frac{\pi}{3}$

3. Sketch the graphs of the following on the interval $[0,2\pi]$

$y = 2\sin 3x$

$y = \tan(x + \pi/4)$

$y = 3\sec x$

4. Solve using trig IDs on the interval $[0,2\pi]$

$2\sin x \cos x = \frac{1}{2}$

$\sin^2 x + \cos x = 1$

5. Use trig IDs to show the following

$$\frac{1+\sin x}{\cos x} = \frac{\cos x}{1-\sin x}$$

$\tan^2 x + 1 = \sec^2 x$ [Hint: use the fact that $\sin^2 x + \cos^2 x = 1$]

6. True or false?

$\sin x + \cos x = 1$

Whenever sin x has a minimum or maximum, cos x has a zero.

The graph of y = tan x has a period of π.

Since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, it follows that $\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}$

Calculator

7. Solve on $[0,2\pi]$

$\sin(x) = 0.3$

$\cot(x) = -2$

$\sec(x) = 0.3$

Answers

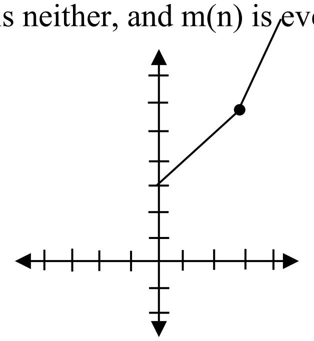
Notice that your answer may be in a different form from the given answer but still be correct.

Linear Functions

1. a. $y-5= 2(x-1)$ b. $y = 2x$ c. $x=6$ d. $y = \frac{b}{a-c}(x-c)$
2. Hint: $f(0)$ is the y-intercept.
3. (b)
4. (d)
5. (b)
6. a. Hint: use the definition of parallel lines. b. Hint: find the equation using point-slope and rearrange.
7. $y = -2x + \frac{1}{2}$
8. $V(t) = -1700t + 15000$; V(t) = 0 when t = 8.8235, which is in late Sept of 2020.
9. $C(F) = \frac{5}{9}(F - 32) = 0.5556F - 17.7778$

Functions

1. Confirm on graphing calculator
2. p(x): domain = {x : x□2}, range = {p: p□0} q(w): domain = [-1,1], range = [0,1] H(t): domain = [-4,4), range = (-2, 4]
3. g(x) is odd, h(x) is neither, and m(n) is even. Use the "-x test" to justify each answer.
4.

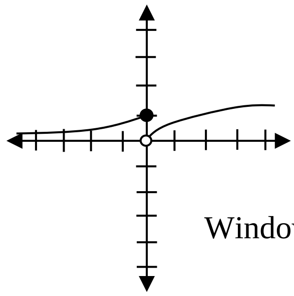


Window: [-4,4]x[-2,7]

5. $v(t) = \begin{cases} □\frac{1}{2}t + 2 & t □ 2 \\ □1 & t = 2 \end{cases}$

$-\frac{1}{2}t + 1 \quad t$ □ 2

6. e.
7. graph:



Window: [-4,4]x[-5,4]

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8. domain: {x: x □ 0}, range: [-1,1]

9. (a) $f^{-1}(x) = \frac{x-5}{3}$

   (b) not invertible

   (c) not invertible

   (d) □$(p) = \cos^{-1}(p)$

   (e) $w(r) = \ln r$

Polynomial Functions

1. Yes. The formula $y = mx + b$ is a polynomial of degree 1 (or degree 0 if m = 0, or undefined degree if $m = 0$ and $b = 0$).
2. Confirm graphs on calculator
3. $H(t) = -\frac{1}{2}(t+2)^2(t-2)$ and □$(r) = \frac{1}{16}(r+2)^2(r-2)^2$ [similar answers are possible]
4. x = 4 or x = 3/2.
   x = 0 or x = 2
5. x = -0.7915 or x = 0.7915 x = 0.4508 or x = 1

Exponential and Logarithmic Functions

1. b.
2. $y = 4(3^x)$

   $y = -2\left(\frac{1}{2}\right)^x$ OR $y = -2\left(2^{-x}\right)$
3. Confirm on calculator
4. x = ln(3/4) $x = \frac{1}{2}\ln 6$

$x = \sqrt{\ln 5}$

5. Confirm on calculator
6. $x = e^3 - 1$ x = -8/9
7. (-0.8326, 0.8326) x = 0.1550 or x = 2

$\{x: 0 < x < 1\}$

Trig Function Problems

1. $\frac{1}{\sqrt{2}}, \frac{\sqrt{}}{2}, -1,$ 2
2. $x =$ □/6, 5□/6

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[0.3□/4)□(5□/4, 2□] x

= 3□/4, 7□/4 x =□/6,

5□/6

$x = \sqrt[3]{2}$

3. Confirm on calculator
4. $x =$ □, 5□, 13□, 17□ 12 12 12 12

$x = \frac{□}{2}, \frac{3□}{2}, 0, 2□$

5. Hint for the first one: multiply the left side by $\frac{1 - \sin x}{1 - \sin x}$.
6. False, true, true, false.
7. $x = 0.3047, 2.8369$ $x = 2.6779, 5.8195$ No solution